A Real Generalized Trisecant Trichotomy

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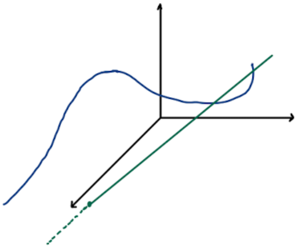
Classical and Generalized Trisecant Lemma

Classical Trisecant Lemma:
A general chord of a non-degenerate algebraic space curve is not a trisecant.



Generalized Trisecant Lemma[1] **(curves in $\mathbb{P}^3_{\mathbb{C}} \Rightarrow$ non-degenerate variety $X \in \mathbb{P}^{N-1}_{\mathbb{C}}$):**

If $P_1, \ldots, P_n$ are general points on X, $W = \text{Span}\{P_1, \ldots, P_n\}$ and we have $\dim X + \dim W < N - 1$, then $X \cap W = \{P_1, \ldots, P_n\}$.

1.Luca Chiantini and Ciro Ciliberto. Weakly defective varieties. Trans. Amer. Math. Soc., 354(1):151–178, 2002.

A Trichotomy from the Generalized Trisecant Lemma

Theorem

Let $X \subseteq \mathbb{P}_{\mathbb{C}}^{N-1}$ *be an irreducible, reduced, non-degenerate projective variety. Let* $P_1, \ldots, P_n$ *be general points on* X *and let* $W = \mathrm{Span}\{P_1, \ldots, P_n\}$. *Then,*

(a) *(Generalized Trisecant Lemma)*
If $\dim X + \dim W < N - 1$, *then* $X \cap W = \{P_1, \ldots, P_n\}$.

(b) *If* $\dim X + \dim W = N - 1$, *then* $\deg X \geq n$.
When $\deg X > n$, $X \cap W \supsetneq \{P_1, \ldots, P_n\}$.
When $\deg X = n$, $X \cap W = \{P_1, \ldots, P_n\}$ *and* X *is a variety with minimal degree; It can be a quadric hypersurface, a cone over the Veronese surface, or a rational normal scroll.*

(c) *If* $\dim X + \dim W > N - 1$, *then* $X \cap W \supsetneq \{P_1, \ldots, P_n\}$.

Question:
What is the analogue of the trichotomy over $\mathbb{R}$?

From $\mathbb{C}$ to $\mathbb{R}$

- $X \subseteq \mathbb{P}^{N-1}_{\mathbb{C}}$ smooth irreducible non-degenerate projective variety of dimension d defined by real coefficients polynomials with a real smooth point on it.
- $P_1, \dots, P_n \in X$ general real points that span W.

Question: When does $(X \cap W)_{\mathbb{R}} = \{P_1, \dots, P_n\}$?

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- When $\dim X + \dim W > N - 1$, $X \cap W$ has positive dimension and it contains smooth real point so $(X \cap W)_{\mathbb{R}}$ has positive dimension, so $(X \cap W)_{\mathbb{R}} \supsetneq \{P_1, \ldots, P_n\}$.
- When $\dim X + \dim W < N - 1$, $X \cap W = \emptyset$ from the generalized Trisecant Lemma, so $(X \cap W)_{\mathbb{R}} = \{P_1, \ldots, P_n\}$.

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- The difficult case is when when $\dim X + \dim W = N - 1$.
- $X \cap W$ contains finitely many points.
- Need to understand the set

$$\mathcal{N}(X) := \text{the set of possible numbers of real points in } X \cap W, \\ W \text{ real linear space}, \dim W + \dim X = N - 1, W \cap X \text{ transversely}$$

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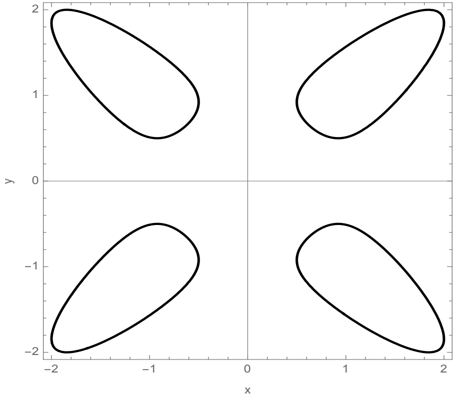
- When $\dim X + \dim W = N - 1$ and $\deg X \not\equiv n \mod 2$, $(X \cap W)_{\mathbb{R}} \supsetneq \{P_1, \ldots, P_n\}$.
- When $\dim X + \dim W = N - 1$ and $\deg X \equiv n \mod 2$,
  - (i) If $n \notin \mathcal{N}(X)$, $(X \cap W)_{\mathbb{R}} \supsetneq \{P_1, \ldots, P_n\}$;
  - (ii) If $n \in \mathcal{N}(X)$ and there is $n' \in \mathcal{N}(X)$ with $n' > n$, there is a nonempty proper open set of real linear spaces such that $(X \cap W)_{\mathbb{R}} = \{P_1, \ldots, P_n\}$;
  - (iii) If $\mathcal{N}(X)_{\max} = n$, $(X \cap W)_{\mathbb{R}} = \{P_1, \ldots, P_n\}$.

Visualize $\mathcal{N}(X)$

The Edge quartic C defined by

$$25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2) = 0. \quad (1)$$

$\mathcal{N}(C) = \{0, 2, 4\}$.



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The Edge quartic C defined by

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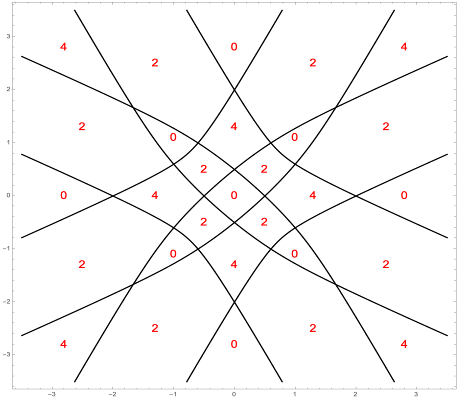

Characterizing the Possible Numbers of Real Solutions

Recall that

$$\begin{aligned}\mathcal{N}(X) :=& \text{the set of possible numbers of real points in } X \cap W, \\ & W \text{ real linear space}, \dim W + \dim X = N - 1, W \pitchfork X \text{ transversely}\end{aligned}$$

We proved the following characterization of $\mathcal{N}(X)$.

Theorem (Kristian Ranestad, Anna Seigal, and KW 2024)

Let $X \subseteq \mathbb{P}_{\mathbb{C}}^{N-1}$ be a smooth real projective variety of dimension d with a smooth real point. Then $\mathcal{N}(X)$ satisfies

(i) $\mathcal{N}(X) = \{k : \mathcal{N}(X)_{\mathrm{min}} \leq k \leq \mathcal{N}(X)_{\mathrm{max}}, k \equiv \deg X \mod 2\}$;

(ii) $N - \dim X \leq \mathcal{N}(X)_{\mathrm{max}} \leq \deg X$.

Regions of fixed number of real solutions in the Grassmannian

We define $\mathcal{U}_k \subseteq \mathrm{Gr}(N-d-1, N-1)_{\mathbb{R}}$ to be the set of $(N-d-1)$-dimensional linear spaces in $\mathbb{P}_{\mathbb{R}}^{N-1}$ that intersect X transversely in exactly k real intersection points.

- The set $\mathcal{U}_k$ is open in $\mathrm{Gr}(N-d-1, N-1)_{\mathbb{R}}$.

2.Sturmfels, Bernd. "The Hurwitz form of a projective variety." Journal of Symbolic Computation 79 (2017): 186-196.

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- The set $B := \text{Gr}(N-d-1, N-1)_{\mathbb{R}} - \bigcup_{k \in \mathcal{N}(X)} \mathcal{U}_k$ is an irreducible hypersurface defined by the Hurwitz form[2]. It contains linear spaces in $\mathbb{P}^{N-1}_{\mathbb{R}}$ that intersect X at some point with multiplicity at least two or in some positive dimension variety.

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- If $\mathcal{U}_i, \mathcal{U}_j$ are smoothly adjacent, meaning that $\overline{\mathcal{U}_i} \cap \overline{\mathcal{U}_j}$ contains some smooth point of the boundary $B := \mathrm{Gr}(N-d-1, N-1)_{\mathbb{R}} - \bigcup_{k\in\mathcal{N}(X)} \mathcal{U}_k$, then $i = j+2$ or $i = j-2$.

2.Sturmfels, Bernd. "The Hurwitz form of a projective variety." Journal of Symbolic Computation 79 (2017): 186-196.

$\mathcal{N}(X)_{\text{min}}$ and $\mathcal{N}(X)_{\text{max}}$ for Segre-Veronese varieties

- Segre-Veronese varieties parametrize partially symmetric rank one tensors $\mathbf{a}_1^{\otimes d_1} \otimes \ldots \otimes \mathbf{a}_k^{\otimes d_k} \in (\mathbb{R}^{m_1+1})^{\otimes d_1} \otimes \cdots \otimes (\mathbb{R}^{m_k+1})^{\otimes d_k}$ up to scale. They are $\mathbb{P}_{\mathbb{C}}^{m_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{m_k}$ embedded via $O(d_1, \ldots, d_k)$.
- If $k = 1$, rank-1 symmetric tensors $=$ Veronese varieties.
- If $d_1, \ldots, d_k = 1$, usual rank-1 tensors $=$ Segre varieties.

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- If $k = 1$, rank-1 symmetric tensors = Veronese varieties.
- If $d_1, \ldots, d_k = 1$, usual rank-1 tensors = Segre varieties.

Theorem (Kristian Ranestad, Anna Seigal, and KW 2024)

For X the Segre-Veronese variety of $\mathbb{P}_{\mathbb{C}}^{m_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{m_k}$ embedded via $O(d_1, \ldots, d_k)$,

(i) $\mathcal{N}(X)_{\max} = \deg X = \frac{(m_1+\ldots+m_n)!}{m_1!\cdots m_n!} \prod_{i=1}^{n} d_i^{m_i}$;

(ii) *When at least two of $m_1, \ldots, m_n$ are odd, then $\mathcal{N}(X)_{\min} = 0$;*

(iii) *When at least one of $d_1, \ldots, d_n$ is even, then $\mathcal{N}(X)_{\min} = 0$.*

$\mathcal{N}(X)_{\text{min}}$ and $\mathcal{N}(X)_{\text{max}}$ for Segre-Veronese varieties (continued)

More things can be said for small Segre varieties.

Theorem (Kristian Ranestad, Anna Seigal, and KW 2024)

For Segre varieties $\mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$, we have

(i) $\mathcal{N}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}})_{\text{min}} = 0$ *if n is odd and* $\mathcal{N}(X)_{\text{min}} = 1$ *if n is even.*

(ii) $\mathcal{N}(\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}})_{\text{min}} \leq \mathcal{N}(\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}})_{\text{min}} + \mathcal{N}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}})_{\text{min}}$

(iii) $\mathcal{N}(\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}})_{\text{min}} \leq \lfloor \frac{n-2}{2} \rfloor$.

Open Question: Let X be the Segre-Veronese variety of $\mathbb{P}^{m_1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{m_k}_{\mathbb{C}}$ embedded via $O(d_1, \ldots, d_k)$. What is $\mathcal{N}(X)_{\text{min}}$ when $d_1, \ldots, d_n$ are all odd and there is at most one odd integer among $m_1, \ldots, m_n$?

Application: Independent Component Analysis

- ICA writes observed variables as linear mixtures of independent sources, i.e.

$$\mathbf{x} = A\mathbf{s},$$

  where $\mathbf{s} = (s_1, \ldots, s_J)^\mathsf{T}$ is independent sources, $\mathbf{x} = (x_1, \ldots, x_I)^\mathsf{T}$ is the observed variables, and $A \in \mathbb{R}^{I \times J}$ is an unknown mixing matrix.
- The ICA model is identifiable if the mixing matrix A can be uniquely recovered, up to some equivalence. A matrix $A \in \mathbb{R}^{I \times J}$ is identifiable if for any vector of source variables $\mathbf{s} = (s_1, \ldots, s_J)$ with at most one Gaussian source, one can recover A uniquely up to some equivalence.

Identifiability has an algebraic geometric criterion.

Theorem (KW and Anna Seigal, 2024)

Fix $A \in \mathbb{R}^{I \times J}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_J$ and no pair of columns collinear. Then A is identifiable if and only if the linear span of $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any real matrix $\mathbf{b}^{\otimes 2}$ unless $\mathbf{b}$ is collinear to $\mathbf{a}_j$ for some $j \in \{1, \ldots, J\}$.

Application: Independent Component Analysis (continued)

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When does a linear space spanned by J real points in the second Veronese embedding of $\mathbb{P}_{\mathbb{C}}^{I-1}$, intersect the second Veronese in exactly these J real points?

Application: Independent Component Analysis (continued)

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Theorem (KW and Anna Seigal, 2024)

Let $A \in \mathbb{R}^{I \times J}$ be generic. Then

(i) *If $J \leq \binom{I}{2}$ or if $(I, J) = (2, 2)$ or $(3, 4)$, then A is identifiable;*

(ii) *If $J = \binom{I}{2} + 1$, where $I \geq 4$ and $I \equiv 2, 3 \mod 4$, then there is a positive probability that A is identifiable and a positive probability that A is non-identifiable;*

(iii) *If $J > \binom{I}{2} + 1$ or if $J = \binom{I}{2} + 1$ and $I \equiv 0, 1 \mod 4$, then A is non-identifiable.*

Thank you!

See arXiv:2409.01356 for more details.

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